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Fractional Integrals Operators and Their Applications in Mathematics and Other Sciences

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We certify that we have read the present work and that in our opinion it is fully adequate in scope and quality as thesis towards the partial fulfillment of the Master Degree requirements in

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Declaration

I certify that all the materials in this thesis are well referenced and its contents is not submitted, in any other place, to get another certificate.

The contents of this thesis reflect my own personal views, and are not necessarily endorsed by AL-Hussein Bin Talal University.

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Dedication

To
My beloved parents who keep holding a candle in front of me during the whole my life.

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Abstract

The fractional calculus is a theory of integrals and derivatives of arbitrary real order. In recent years considerable interest in fractional calculus has been stimulated by the applications, which can be found in different areas of applied sciences like physics and engineering.

This thesis is simply a research on fractional integrals, a topic that found to be fascinating. The reader should be delighted by introduction and short history of the topic in chapter one. In this thesis we studied fractional order integral and derivative, we have introduced basic definition of many fractional integral and derivative operators, many important properties of these operators have been provided and we have worked on its proof, this will give a better understand of these operators, illustrated by some examples. Also, we provide many applications of these operators which recently used by researchers and focus on some of these applications.

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List Of Abbreviations

D_*^α	Caputo fractional derivative.
D_a^α	Riemann-Liouville fractional derivative.
$\beta(x, y)$	beta function.
$\Gamma(\alpha)$	gamma function.
erf	the error function.
$erfc$	complementary error function.
$\delta(v, t)$	incomplete gamma function.
$\Gamma_k(x)$	K-gamma function.
$\beta_k(x, y)$	K-beta function.
I^α	Liouvill fractional integral.
${}^pI^\alpha$	Generalized fractional integral.
${}^pD^\alpha$	Generalized fractional derivative.
J^α	Hadamard fractional integrals.
W_k^α	K-Weyle fractional integral.
I_a^α	Riemann-Liouville fractional integrals.
${}_k^qJ_a^\alpha$	(k,q)-Riemann-Liouville fractional integrals.
D^α	Liouville fractional derivatives.
Lp	The set of all Lebesgue measurable function.
${}_aI_b^\alpha$	The Local fractional integrals.

Introduction

Fractional calculus is a branch of mathematics deals with the theory of integrals and derivatives of order arbitrary real or even complex number. This branch of science has received great attention from researchers since few decades ago because of it's important applications in a lot of scientific problems such as physical applications, chemical applications and dynamical control systems, electronic networks and many more.

Understanding of definitions and the use of fractional calculus will be made more clearly by quickly discussing some necessary and relatively simple mathematical definitions that will rise in the study of these concepts. These are the gamma function, The beta function, Dirichlet's formula, Fubini's theorem, K-Gamma function, K-beta function, Leibniz's rule, incomplete gamma function and the incomplete beta function are addressed in the following subsections.

1.1. Historical Background

The Leibniz's letter to L'Hospital is considered the beginning of the fractional calculus science. The notation of non-integer order $\frac{1}{2}$ is discussed. Leibniz's wrote: " Although infinite series and geometry are distinct relations, infinite series admits only the use of exponents that are positive and negative integer and does not yet use of fractional exponents,this is an apparent paradox from which, one day, useful consequences will be drawn". In 1730 Leonhard Euler extended the integer order derivative of a monomial to non-integer order in terms of the Gamma function. In late 1800s Joseph Liouville similarly extended the derivative formula acting on the exponential to non-integer order. In that sense many first fractional derivatives of functions came from recursive relationships[**1**].

Following L'Hospital's and Leibniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace, Riemann, Liouville, Weyl, Hadamard, Caputo were all interested in fractional calculus and its mathematical consequences. they used their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that has been popularized in the world of fractional calculus is the Riemann-Liouville definition. While the shear number of actual definitions are no doubt as numerous as the people who study this field[**2**]. Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However it is in the past 100 years that the most definitions of fractional calculus and scientific applications have been studied .

Recently, there are many published books,researches, proceedings, and even special issues of journals that refer to the applications of fractional calculus in several scientific areas including special functions, control theory, chemical physics, stochastic processes, anomalous diffusion, rheology and many more. Several special issues appeared in the last decade which contain selected and improved papers presented at conferences and advanced schools, concerning various applications of fractional calculus. Since several years, there exist many international journals devoted almost exclusively to the subject of fractional calculus such as: Journal of fractional Calculus (Editor-in-Chief: K. Nishimoto, Japan) started in 1992, and Fractional Calculus and Applied Analysis (Managing Editor: V. Kiryakova, Bulgaria) started in 1998. Journal of Fractional Dynamic Systems started in 2010. We believe that the volume of research in the area of fractional calculus will continue to grow in the forthcoming years and that it will constitute an important tool in the scientific progress of mankind[**3**].

In fact, fractional calculus asserts that orders of integral or derivative operators can be arbitrary numbers, for instance, one could calculate the half order integral or the $\sqrt{5}^{th}$ order of an analytic function. However, what if we want to integrate our function $f(x)$ to the half order ?, how could we define our operations?, would our results have a meaning or an application comparable to that of the familiar integer order operations?

1.2. Literature Reviews

The definition of some special functions and their properties will introduced.

1.3. Thesis Objectives

The main topic in this research is fractional integral. This thesis is intend to:

(1) Collect the results of many studies of the fractional integrals operators in a well-organized study.

(2) Prove and understand many of the important properties of the fractional integrals operators.

These will provide the researchers and interested people to compare between these results and to create new methods and models for applicable problems in mathematics and other branches of science.

1.4. Thesis Organization

This thesis consists of five chapters. Chapter One, is the introduction of this research. Which includes, some of the earlier studies, the main objectives, and the research organization.

In Chapter Two, we give the history of the fractional calculus, and we present some basic definitions and properties that are used in the definitions of fractional calculus.

Chapter Three, presents fractional calculus operators, and we provide some definitions and give some examples, and prove some properties of these operators and how the operators represent some special function.

Next, in Chapter Four, we have shown the uses of fractional calculus and referred to the application of fractional calculus in other sciences.

In the last chapter, the conclusions of this research were concluded and discussed.

Special Functions

2.1. Introduction

Before looking at the definition of the fractional integral or derivative, we will state some basic definitions, notations and results that will be used through this thesis. These include some of special functions, such as, the Gamma function, Beta function, Dirichlet's formula, Fubini's theorem, K-gamma function, K-beta function and incomplete gamma function.

2.2. Some Special Functions

2.2.1. Gamma function[1]. The gamma function has very important applications in science. it is a continuous extension to the factorial concept of positive integers, which is only defined for the nonnegative integers. While there are other continuous extensions to the factorial function, the gamma function is the only function that is convex for positive real number.

The gamma function $\Gamma(z)$, for $z > 0$ is given by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds, \ z \in \mathbb{R}^+. \qquad \textbf{(2.2.1)}$$

The gamma function has many properties **[1]**, such as the recursion relation :

(1) $\Gamma(z+1) = z\Gamma(z), \ z > 0$
It's clear $\Gamma(1) = 1$ and for $z = 1, 2, 3, \cdots$ we have

$$\Gamma(2) = 1\Gamma(1) = 1! = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2 \cdot 1! = 2!$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2! = 3! \ , \quad \text{and so on.}$$

(2) $\Gamma(n+1) = n\Gamma(n) = n \cdot (n-1)! = n!, n \in N.$

(3) $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$, for $\alpha > 0$

(4) $\Gamma(\alpha) = \dfrac{\Gamma(\alpha + 1)}{\alpha}$, for $\alpha \in \mathbb{R}$

(5) $\Gamma\left(n + \frac{1}{2}\right) = \dfrac{(2n-1)!}{2^{2n-1}(n-1)!} \sqrt{\pi}, n \in N.$

(6) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

(7) $\Gamma(n)$ is undefined for negative integers.

Example 1. (1) $\Gamma\left(\frac{9}{2}\right)=\Gamma\left(4+\frac{1}{2}\right)=\frac{7!}{2^7 3!}\sqrt{\pi}=\frac{105}{16}\sqrt{\pi}$.

(2)

$$\begin{aligned}\Gamma\left(\tfrac{23}{3}\right) &= \left(\tfrac{20}{3}\right)\Gamma\left(\tfrac{20}{3}\right)\\ &= \left(\tfrac{20}{3}\right)\left(\tfrac{17}{3}\right)\Gamma\left(\tfrac{17}{3}\right)\\ &= \left(\tfrac{20}{3}\right)\left(\tfrac{17}{3}\right)\left(\tfrac{14}{3}\right)\left(\tfrac{11}{3}\right)\left(\tfrac{8}{3}\right)\left(\tfrac{5}{3}\right)\Gamma\left(\tfrac{5}{3}\right)\\ &= \tfrac{2094400}{729}\Gamma\left(\tfrac{5}{3}\right)=\tfrac{2094400}{729}\Gamma(1.67)=\tfrac{2094400}{729}(0.90330)\\ &= 2595.1598\end{aligned}$$

(3)

$$\Gamma\left(-\tfrac{5}{6}\right)=\frac{\Gamma\left(\frac{-5}{6}+1\right)}{-\frac{5}{6}} = -\tfrac{6}{5}\Gamma\left(\tfrac{1}{6}\right)=-\tfrac{6}{5}\frac{\Gamma\left(\frac{7}{6}\right)}{\frac{1}{6}}=-\tfrac{36}{5}\Gamma\left(\tfrac{7}{6}\right)=-\tfrac{36}{5}(0.927733333333)=-0.67968$$

Table 2.2.1. Gamma function, $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\cdot dx$. See [**1**]

α	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	1.00000	0.99433	0.98884	0.98355	0.97844	0.97350	0.96874	0.96415	0.9597	0.95546
1.1	0.9515	0.94740	0.94359	0.93993	0.93642	0.93304	0.92980	0.92670	0.92373	0.92089
1.2	0.91817	0.91558	0.91311	0.91075	0.90852	0.90640	0.90440	0.90250	0.90072	0.89904
1.3	0.89747	0.89600	0.89464	0.89338	0.89222	0.89115	0.89018	0.88931	0.88854	0.88785
1.4	0.88726	0.88676	0.88636	0.88604	0.88581	0.88566	0.88560	0.88563	0.88575	0.88595
1.5	0.88623	0.88659	0.88704	0.88757	0.88818	0.88887	0.88964	0.9049	0.89142	0.89243
1.6	0.89352	0.89468	0.89592	0.89724	0.8964	0.90012	0.90167	0.90330	0.90500	0.90678
1.7	0.90864	0.91057	0.91258	0.91467	0.91683	0.91906	0.9217	0.276	0.92623	0.92877
1.8	0.93138	0.93408	0.93685	0.93969	0.94261	0.94561	0.94869	0.95184	0.95507	0.95838
1.9	0.96177	0.96523	0.96577	0.97240	0.97610	0.97988	0.98374	0.98768	0.99171	0.99581

2.2.2. Beta function[1]. The Beta function denoted by $\beta(x,y)$ is a multi variable function given by :

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \text{ for } x,y \in \mathbb{R}^+ \tag{2.2.2}$$

If we set $t = cos^2\theta$ in (**2.2.2**) we find that

$$\begin{aligned} \beta(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \\ &= -2\int_{\frac{\pi}{2}}^0 cos^{2x-2}\theta(1-cos^2\theta)^{y-1}cos\theta\ sin\theta\ d\theta \\ &= 2\int_0^{\frac{\pi}{2}} cos^{2x-1}\theta sin^{2y-1}\theta\ d\theta \end{aligned} \tag{2.2.3}$$

Example 2.

$$\begin{aligned} \beta(2,3) &= \int_0^1 t(1-t)^2dt = \int_0^1 (t-2t^2+t^3)dt \\ &= \left(\tfrac{t^2}{2} - \tfrac{2}{3}t^3 + \tfrac{t^4}{4}\right)\Big|_0^1 = \tfrac{1}{12} \end{aligned}$$

The following are some properties of beta function:

(1) An important property[**1**] connecting the Gamma and Beta function is:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \text{ for } x,y \in \mathbb{R}^+ \tag{2.2.4}$$

Example 3. (ii)

$$\beta(6,4) = \frac{\Gamma(6)\Gamma(4)}{\Gamma(10)} = \frac{5!\cdot 3!}{9!} = \frac{1}{504}$$

(ii)

$$\beta(\frac{1}{2},\frac{1}{2}) = \pi$$

(iii)

$$\beta(1.7,1.8) = \frac{\Gamma(1.7)\Gamma(1.8)}{\Gamma(3.5)} = \frac{(0.90864)(0.93138)}{\Gamma(3+\frac{1}{2})} = \frac{0.8462891232}{3.3233509704} = 0.2546493376$$

(2) $\beta(x,y)$ is a symmetric function. i.e

$$\beta(x,y) = \beta(y,x), \text{ for } x>0, y>0$$

Proof.

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

put $u=1-t$

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt=\int_1^0 (1-u)^{x-1}u^{y-1}(-du)=\int_0^1 u^{y-1}(1-u)^{x-1}du=\beta(y,x)$$

□

2.2.3. The error function[4]. The error function is given by:

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\ ,\ x\in\mathbb{R} \qquad \textbf{(2.2.5)}$$

Example 4.

$$\begin{aligned}
erf(\infty) &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt \\
&\text{put} \quad u=t^2 \\
erf(\infty) &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u}\,\frac{1}{2}\,u^{-\frac{1}{2}}\,du \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-u}\,u^{1-\frac{1}{2}}\,du=\frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2})=1 \\
\text{and}\quad erf(0) &= \frac{2}{\sqrt{\pi}}\int_0^0 e^{-t^2}\,dt=0
\end{aligned}$$

2.2.4. The complementary error function. The complementary error function is given by [**4**]:

$$erfc(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt=1-erf(x) \qquad \textbf{(2.2.6)}$$

2.2.5. Incomplete Gamma Function[5]. The incomplete gamma function for $v>0$ is given by

$$\delta(v,t)=\int_0^t x^{v-1}e^{-x}dx \qquad \textbf{(2.2.7)}$$

Clearly, $\delta(v,t)\to\Gamma(v)$ as $t\to\infty$.

The properties of $\delta(v,t)$ are listed in many references. In particular, the following properties[**5**] are needed:

(1) $\delta(v+1,t) = v\delta(v,t) - t^v e^{-t}$

(2) $\delta(1,t) = 1 - e^{-t}$

(3) $\delta\left(\frac{1}{2},t\right) = \sqrt{\pi}\; erf\left(\sqrt{t}\right)$.

Prove the property (3)

Proof.

$$\delta(\tfrac{1}{2},t) = \int_0^t x^{\frac{-1}{2}}\; e^{-x}\; dx$$

put $x = s^2$, we obtain

$$\begin{aligned}\delta(\tfrac{1}{2},t) &= \int_0^{\sqrt{t}} \frac{1}{\sqrt{s^2}}\; e^{-s^2}\; 2\; s\; ds \\ &= 2\left(\frac{\sqrt{\pi}}{2} erf(\sqrt{t})\right) = \sqrt{\pi}\; erf(\sqrt{t}).\end{aligned}$$

□

2.2.6. K-gamma function [6]. For a positive real number k and a complex number x with $\mathrm{Re}(x) > 0$, the K-gamma function is defined by the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt \qquad (\mathbf{2.2.8})$$

This function has the following properties[6]:

(1) $\Gamma_k(k) = 1$.

(2) $\Gamma_k(x+k) = x\Gamma_k(x)$.

(3) $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$.

(4) $\Gamma_k(x) = a^{\frac{x}{k}} \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\; dt$.

Where a and x are real numbers.

2.2.7. K-beta function[6]. The K-beta function is given by :

$$\beta_k(x,y) = \int_0^\infty t^{x-1}(1+t^k)^{-\frac{x+y}{k}} dt \qquad (\mathbf{2.2.9})$$

The K-beta function has the following properties[6]:

(1) $\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt$

(2) $\beta_k(x,y) = \frac{1}{k}\beta(\frac{x}{k},\frac{y}{k})$

Theorem 2.2.1. *(Fubini's Theorem)***[4]**

let $f(x,y)$ be a continuous function on $R = \{(x,y) : x \in [a,b], y \in [c,d], a,b,c,d \in \mathbb{R}\}$ then

$$\int\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

notice that if $f(x,y) = g(x)\ h(y)$ then

$$\begin{aligned}
\int\int_R f(x,y)\ dA &= \int\int_R g(x)h(y)dA \\
&= \int_c^d \int_a^b g(x)h(y)dxdy \\
&= \int_c^d h(y)\left(\int_a^b g(x)dx\right)dy \\
&= \int_a^b g(x)dx \int_c^d h(y)dy
\end{aligned}$$

Example 5. (1)

$$\begin{aligned}
&\int\int_R 6xy^2\ dA, \qquad R = [2,4] \times [1,2] \\
&= \int_2^4 \int_1^2 6xy^2\ dy\ dx \\
&= \int_2^4 (2xy^3)\Big|_{y=1}^{y=2} dx = 84
\end{aligned}$$

(2)

$$\begin{aligned}
&\int\int_R (x^2y^2 + \cos(\pi x) + \sin(\pi y))dA \qquad , R = [-2,-1] \times [0,1] \\
&= \int_0^1 \int_{-2}^{-1} (x^2y^2 + \cos(\pi x) + \sin(\pi y))dx\ dy \\
&= \int_0^1 (\frac{1}{3}x^3y^2 + \tfrac{1}{\pi}\sin(\pi x) + x\sin(\pi y))\Big|_{x=-2}^{x=-1} dy \\
&= \int_0^1 (\frac{7}{3}y^2 + \sin(\pi y))dy = \left(\frac{7}{9}y^3 - \frac{1}{\pi}\cos(\pi y)\right)\Big|_{y=0}^{y=1} = \frac{7}{9} + \frac{2}{\pi}
\end{aligned}$$

(3)

$$\int\limits_R\int x\cos^2 y\, dA,\ R=[-2,3]\times[0,\frac{\pi}{2}]$$
$$=(\int\limits_{-2}^{3} x\, dx)(\int\limits_{0}^{\frac{\pi}{2}} \cos^2 y\, dy)=\frac{5\pi}{8}$$

2.2.8. Dirichlet's Formula[4]. Dirichlet's formula is an application of Fubini's theorem in real analysis, when $f(x,y)=(t-x)^r(x-y)^s h(x,y)$.

let f be a continuous function on the xy-plane, and let μ , $v\in R^+$, then:

$$\int\limits_0^t (t-x)^{\mu-1}\, dx\int\limits_0^x (x-y)^{v-1}h(x,y)\, dy=\int\limits_0^t dy\int\limits_y^t (t-x)^{\mu-1}(x-y)^{v-1}h(x,y)dx$$

2.2.9. Fundamental theorem of calculus.

$$\text{If}\qquad F(x)=\int_{\alpha(x)}^{\beta(x)} f(x,t)\, dt$$

then

$$\frac{d}{dx}\,F(x)=f(x,\beta(x))\,\frac{d}{dx}\beta(x)-f(x,\alpha(x))\,\frac{d}{dx}\alpha(x)+\int_{\alpha(x)}^{\beta(x)}\frac{\partial f(x,t)}{\partial x}\,dt\,. \qquad (\mathbf{2.2.10})$$