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APPLICATIONS OF REPRODUCING KERNEL METHOD FOR SOLVING SYSTEM OF BOUNDARY VALUE PROBLEMS

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DECLARATION

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Dedication

To the soul of my father

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List of Abbreviations

Number	Abbreviation	Meaning
1	RKHSM	Reproducing kernel Hilbert space method
2	BVPs	Boundary value problems
3	Abs. Cts.	Absolutely continuous
4	$\mathbb{R}$	Set of real numbers
5	$\mathbb{C}$	Set of complex numbers
6	$\mathbb{N}$	Set of natural numbers
7	$u_n(x)$	Approximate solution
8	$u(x)$	Exact solution
9	$\mathcal{H}$	Hilbert space
10	$W_2^3[a,b]$	Sobolev space of order 3 over $[a,b]$
11	L_2	Lebesgue space (L_2-space)
12	$\{\psi_i\}_{i=1}^{\infty}$	Orthogonal basis
13	$\{\bar{\psi}_i(t)\}_{i=1}^{\infty}$	Orthonormal basis
14	$\{t_i\}_{i=1}^{\infty}$	Dense subset of $[a,b]$
15	β_{ik}	Orthogonalization coefficients
16	$K_s(t)$	Reproducing kernel function
17	L_y	Linear differential operator of y
18	$\langle y,y\rangle_W$	Inner product in sense of W_2^3
19	$\|y\|_W$	The norm function on W_2^3
20	$\delta(x)$	Dirac-delta function

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ABSTRACT

In this thesis, an accurate numerical approximated algorithm based on the reproducing kernel Hilbert space (RKHS) approach has been proposed to solve system of second-order boundary value problems (BVPs) associated with nonclassical boundary conditions. The analytical solution has been presented in the form of convergent series with accurately computable structures in reproducing kernel space $W_2^3[0,1]$. The n-term approximation has been obtained and proved to converge uniformly to the analytical solution. The main advantage of the RKHS approach is that it can be directly applied without requiring linearization or perturbation, and therefore, it is not affected by errors associated with discretization. Some numerical examples have been given to demonstrate the computation accurately, effectiveness of the proposed approach. The numerical results obtained showed that the RK method is a powerful tool in finding effective approximated solutions to such systems arising in applied mathematics, physics and engineering.

Introduction

System of differential equations are used to understand the physical, engineering and biological sciences; Differential equations have been developed and become increasingly important in all areas of science and its applications. This subject was originated by Issac Newton (1642-1727) and Gottfried Leibniz (1646-1716) while they are studying Calculus in seventeenth century. After that there was a lot of development in this field by many mathematicians such as the brothers of Jakob (1654-1705), Johann Bernoulli (1667-1748), and Leonhard Euler (1707-1783) who was the first identified the conditions for exactness of first-order differential equation. Also, Joseph-Louis Lagrange (1736-1813) who showed that the general solution of an n-*th* order linear homogenous differential equation is a linear combination of n independent solutions. Later a greatest mathematician Pierre-Simon de Laplace (1749-1827) left his mark by using Laplace transform method which is very useful in solving differential equation.

In nineteenth century mathematicians concern turned in investigation of theoretical questions of existence and uniqueness and development methods which depend on power series expansions, solution by analytical methods. While in twentieth century the development was very fast especially in numerical approximation and geometrical or topological methods appeared for solving nonlinear differential equations.

Furthermore, the subject of boundary value problems (BVPs) plays an important role in various applications in physics, biology, engineering, chemistry. Second order initial value problems solved by many numerical methods but unfortunately, we cannot apply these methods to solve second order boundary value problems. Nonlocal problems are used in several phenomena in Chemistry, industry, Physics we can use modeling the problem as nonlocal problems. However, the differential equations have recently proved to be valuable tools to the modeling of many physical phenomena. This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time.

Moreover, the BVPs emerge in a few part of science and designing applications, including wave equation, amorphous systems, micro-emulsions, polymers, biopolymers, etc. To be helpful in applications, a BVP should be well-posed this implies given the input to the issue that there exists a unique solution, which depends constantly on the input data. Many theorems in the field of ordinary or partial differential equations is committed to demonstrate that such BVPs are in truth well-posed. Surely, any data about the answer of the differential equations might be for the most part indicated at more than one point. Regularly there are two points, which compare physically to the boundaries of some area, so it is a two-point BVP.

The numerical solvability of BVPs has been pursued by many authors. To mention few of them, the monotone method has been applied to solve BVPs as described in [8]. In [9], the authors have developed the Rayleigh-Ritz method. Also, in [10], the authors have provided the Tri-Reduction method to further investigations to class of periodic equations. Recently, the reproducing kernel Hilbert space (RKHS) method for solving both linear and nonlinear forms of BVPs. The aim of this thesis is to establish analytical as well as numerical solutions for class of BVPs subject to nonclassical conditions using the reproducing kernel Hilbert space (RKHS) method. However, consider the following general from of nonlinear system of second order Differential Equation:

$$u''(x) = f(x, u(x), u'(x), v(x), v'(x)),$$

$$v''(x) = g\big(x, u(x), u'(x), v(x), v'(x)\big),$$

subject to the different class of boundary conditions including local, non-local, or non-classical, where $0 \leq x \leq 1$, $f: [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ and $g: [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous linear or nonlinear function, and $u(x), v(x)$ are unknown analytical function to be determined over the domain of interest.

The theory of reproducing kernel method was first used at the beginning of the last century as an unfamiliar solver for the BVPs for harmonic and biharmonic functions. This theory, which was established in the RKHS method in 1950 by Aronszajn, has many important applications in approximation theory, numerical analysis, computational mathematics, image processing, machine learning theory, complex analysis, probability and statistics (Saitoh, 1988; Daniel, 2003; Li and Cui, 2003). The RKHS method has an advantage by constructing an important numerical solution in applied sciences. In the previous years, depending on this theory, extreme work has been done in the numerical solutions of several integral and differential operators' side by side with their theories. The reader is kindly requested to go through (Cui and Lin 2008; AL-Smadi, 2011; Geng, 2009; Al-Smadi et al., 2012; Geng and Cui, 2007; Ye and Geng, 2009; Komashynska and Al-Smadi, 2014; Abu Arqub et al., 2012; Abu Arqub et al., 2015; Altawallbeh et al., 2013) in order to know more details about the RKHS method, including its modification and scientific applications, its characteristics and symmetric kernel functions, and others.

Indeed, the reproducing kernel theory has wide applications in numerical analysis, ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations (IEs), integrodifferential equations (IDEs), probability and statistics, and so forth [8-10]. Recently, the authors in [11-14] have discussed Fredholm IDEs, fourth-order Volterra IDEs, fourth-order mixed IDEs of boundary conditions, and Fredholm-Volterra IDEs of first order by using the reproducing kernel Hilbert space (RKHS) method.

Ordinary differential equations are concerned in finding solutions to many problems in different areas. for instance stability of plane in flight and studying biochemical reactions .the most common applications of ODE`s are the oscillations theory ,dynamical system and automatic control theory.

Series expansions are very important aids in numerical calculations, especially for quick estimates made by hand calculation. Solutions of the system of DEs can often be expressed in terms of series expansions. However, the RKHS is an analytical as well as numerical method depending on the reproducing kernel theory for solving different types of ordinary and partial differential equations. The RKHS method is effective and easy to construct power series solution for strongly linear and nonlinear equations without linearization, perturbation, or discretization.

In this thesis, we extend the application of the RKHS method to provide symbolic approximated solution for class of system of differential equations of second order subject to nonclassical boundary conditions. In this approach, the analytic and the approximate solutions are given in series form in the appropriate Hilbert space $W_2^3[0,1]$. The steps of the solution begins by getting the orthogonal basis from the obtained kernel functions; and the orthonormal basis is constructed in order to formulate and use the solutions with the series form. Convergence analysis for the proposed method is also presented. Furthermore, some test examples are given to demonstrate the validity and applicability of the method.

This thesis contains the following chapters , in Chapter One, we present some fundamental definitions and notations related to functional analysis and reproducing kernel theory as well as the reproducing kernel spaces are presented to construct its reproducing kernel functions. It is worthwhile to point out that the reproducing kernel functions has unique representations. In Chapter Two, we construct and obtain the solution for second order system differential equations associated with non-classical boundary conditions based on the RKHSM that usually provides the solution in terms of a rapidly. Also, the convergent analysis has been discussed. Further, some corollaries and remarks related to reproducing kernel subject have been given. In Chapter Three, some numerical examples have been given to show the performance and generality of the proposed method. The last chapter, Chapter Four has some concluding remarks and future recommendations.

Chapter One

Mathematical Preliminaries

The material in this chapter is basic in certain sense. For the reader's convenience, we present some fundamental concepts, preliminaries and necessary notations about the functional analysis and reproducing kernel theory as well as the reproducing kernel spaces are presented to construct its reproducing kernel functions. These results were collected from Aronszajn 1950, and the references in [20-25].

Definition 1.1: A normed space E is a vector space over a field F with a norm defined on it such that $\|\,.\,\| : E \to [0, \infty)$ satisfies:

1) $\|a\| \geq 0$ with equality iff $a = 0$.
2) $\|\alpha a\| = |\alpha| \|a\|, \ \forall \alpha \in F$.
3) $\|a + b\| \leq \|a\| + \|b\|, \ \forall a, b \in E$. (triangular inequality)
4) $|\langle a, b\rangle| \leq \|a\| \|b\|, \ \forall a, b \in E$. (Cauchy-Schwarz Inequality)
5) If $\langle a, b\rangle = 0$ then $\|a + b\|^2 = \|a\|^2 + \|b\|^2, \ \forall a, b \in E$ (Pythagorean Theorem)
6) $\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a, b\rangle, \ \forall a, b \in R$

Note that $\forall a \in E, \ \|a\| = \sqrt{\langle a, a\rangle}$.

Definition 1.2: A vector space E is said to be an inner product space if there is a mapping of $E \times E$ into a field F that satisfies the following properties:

1) $\langle a + b, c\rangle = \langle a, c\rangle + \langle b, c\rangle, \ \forall a, b, c \in E$.
2) $\langle \alpha a, b\rangle = \alpha\langle a, b\rangle, \ \forall a, b \in E, \ \forall \alpha \in F$.
3) $\langle a, b\rangle = \overline{\langle b, a\rangle}$.
4) $\langle a, a\rangle \geq 0$, with equality iff $a = 0$.

Definition 1.3: A sequence $\{e_n\}_{n=1}^{\infty}$ in a normed space E is said to be convergent sequence if there exists $e \in E$ such that $\lim\limits_{n\to\infty} \|e_n - e\| = 0$. We call e the limit of (e_n).

Definition 1.4: A sequence $\{e_n\}_{n=1}^{\infty}$ in a normed space E is called a Cauchy sequence if $\forall \varepsilon > 0$, there is a positive real number N_ε such that $\|e_n - e_m\| < \epsilon, \forall n, m > N_\epsilon$.

Definition 1.5: A Hilbert space H is an inner product space in which every Cauchy sequence in H converges to some element in H.

Remark 1.1: A Banach space is a complete normed space.

Definition 1.6: Let E, S be two normed spaces. A map $A: E \to S$ is called a linear operator if it satisfies

$$A(\alpha t + \beta r) = \alpha A t + \beta A r, \qquad \forall \alpha, \beta \in \mathbb{R}, \qquad \forall t, r \in E.$$

If $S = \mathbb{R}$, then A is called linear functional.

Definition 1.7: Let E, S be two normed spaces and $A: D(A) \subset E \to S$ be a linear operator. Then A is called a bounded linear operator if $\exists \lambda > 0$ such that $\|Ae\| \le \lambda \|e\|, \ \forall e \in D(A)$.

Definition 1.8: Let $A: \mathcal{D}(A) \longrightarrow S$ be any operator, not necessarily linear, where $\mathcal{D}(A) \subset E$ and E, S are two normed spaces, then the operator A is said to be continuous at $e_0 \in \mathcal{D}(A)$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Ae - Ae_0\| < \varepsilon$ for every $e \in \mathcal{D}(A)$ satisfying $\|e - e_0\| < \delta$.

Remark 1.2: A is continuous iff A is continuous at every $e \in \mathcal{D}(A)$.

Definition 1.9: Let $A: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. Then the Hilbert-adjoint operator A^* of A is the operator $A^*: H_2 \longrightarrow H_1$ such that for all $x \in H_1$ and $y \in H_2$, $\langle Ax, y\rangle = \langle x, A^* y\rangle$.

Remark 1.3: A is self - adjoint iff $A = A^*$.

Definition 1.10: A function $f: [a, b] \longrightarrow \mathbb{R}$ is called absolutely continuous (Abs. C), iff for every positive ε, there exists a positive δ such that for any finite set of disjoint intervals $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ $(\subset [a, b])$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$, then $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$.

Theorem 1.1: (***Fundamental theorem of Lebesgue integral calculus***) let $f:[a,b] \longrightarrow \mathbb{R}$ be a function , then f is absoultly continuos if and only if there is a function $g\epsilon L^1[a,b] : f(x) = f(a) + \int_a^x g(t)dt \,, \forall x\epsilon[a,b]$.

Definition 1.11: Let Ω be a domain in $\mathbb{R}^n$ and let $1 \leq p < \infty$. The class of all measurable functions u defined on Ω for which $\int_\Omega |u(x)|^p dx < \infty$ is denoted by $L^p(\Omega)$, which is a Banach space with respect to the norms $\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$.

Lemma 1.1: Let $A: \mathcal{D}(A) \longrightarrow S$ be a linear operator, where $\mathcal{D}(A) \subset E$ and E, S are normed spaces. Then A is continuous if and only if A is bounded.

Theorem 1.2: (*Riesz's Theorem*) Every bounded linear functional A on a Hilbert space $\mathcal{H}$ can be represented in terms of the inner product, namely $A(x) = \langle x, y \rangle$, where y depends on A, is uniquely determined by A and has norm $\|y\| = \|A\|$.

Definition 1.12: A linear functional f is a linear operator with domain in a vector space X and range in the scalar field F of X , thus $f: X \longrightarrow$ F.

Definition 1.13: let $\mathcal{H}$ be a Hilbert space of a function $f: X \to \mathbb{R}$, defined on a non-empty set X.for a fixed $x\epsilon X$, the map $\delta_x: \mathcal{H} \to \mathbb{R}$, such that $\delta_x(f) = f(x)$ is called the Dirac-evaluation at f. Note that δ_x is bounded if $\exists M > 0$ such that $\|\delta_x f\|_{\mathbb{R}} \leq M\|f\|_{\mathcal{H}}, \ \forall f \in \mathcal{H}$.

The delta function $\delta(x)$ has the property $\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$, where the delta function is define as follows:

$$\delta(x-y) = \begin{cases} 0, & x \neq y \\ \infty, & x = y \end{cases}.$$

Definition 1.14: (Sobolev Spaces) The Sobolev $W_p^m(\Omega)$ space is the space of all locally summable functions $u: \Omega \to \mathcal{R}$ such that, for every multi-index α with $|\alpha| \leq m$, the weak derivative $\partial^\alpha u$ exists and belongs to $L^p(\Omega)$, which is given by:

$$W_p^m(\Omega) = \{u \in L^p(\Omega): \partial^\alpha u \in L^p(\Omega), |\alpha| \leq m\},$$

whereas the norm on $W_p^m(\Omega)$ can be defined as

- $\|u\|_{W_p^m(\Omega)} := \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} = \left(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha u|^p \, dx\right)^{\frac{1}{p}}$ if $1 \leq p < \infty$
- $\|u\|_{W_p^m(\Omega)} := \max_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}^p$ if $p = \infty$.

The following are true :

1. Each Sobolev space $W_p^m(\Omega)$ is a Banach space.
2. If $m = 0$, the space $W_p^0(\Omega) = L^p(\Omega)$.
3. If $p = 2$, the space $W_2^m(\Omega)$ is Hilbert space with the inner product:

$$\langle u, v\rangle_{W_2^m} := \int_\Omega \sum_{|\alpha| \leq m} \partial^\alpha u(x)\, \partial^\alpha v(x)\, dx.$$

However, we'll use the $m-$th order Sobolev space $W_2^m[a, b]$ which is defined as follows:

$$W_2^m[a, b] = \left\{u: u^{(m-1)} \text{ is } ABC,, u^{(m)} \epsilon L^2[a, b]\right\}$$

The space $W_2^m[a, b] \subset L^2[a, b]$. Hence, the properties of $L^2[a, b]$ functions will be applicable to the elements of the space $W_2^m[a, b]$.

The inner product and the norm in the function space $W_2^m[a, b]$ are defined as follows:

for any functions $u(x), v(x) \in W_2^m[a, b]$, we have

$$\langle u, v\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a) v^{(i)}(a) + \int_a^b u^{(m)}(x) v^{(m)}(x) dx,$$

and

$$\|u\|_{W_2^m[a,b]} = \sqrt{\langle u, u\rangle_{W_2^m[a,b]}}.$$

$W_2^m[a,b]$ is sometimes denoted by: $W_2^m[a,b] = \mathcal{K}^m[a,b]$.

Remark 1.4 : The kernel R is called Hermitian if for any finite set of points $\{x_1, x_2, \dots, x_n\} \in E$and any complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, we have $\sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j K(x_i, x_j) \in \mathbb{R}$.

Next, we will introduce the definition of a reproducing kernel Hilbert space.

Definition 1.15: Let $E \neq \emptyset$. A function $R: E \times E \to \mathbb{C}$ is called a reproducing kernel function of, the Hilbert space H of functions, if and only if

a) $R(\,.\,, t) \in H$,
b) $\langle \varphi\,, R(\,.\,, t)\rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

The last condition is called "the reproducing property" as the value of the function φ at the point t is reproduced by the inner product of φ with $R(\,.\,, t)$.

Thus, it is easy to note that $\forall x, y \in X$, $R_x(y) = R(x\,, y) = \langle R_x(.), R_y(.)\rangle$ as a function of y belongs to $\mathcal{H}$.

Now, we will present some basic properties of reproducing kernels functions.

Lemma 1.2: If a reproducing kernel R exists, then it is unique.

Proof: Assume that K is another reproducing kernel, then

$$0 < \|R(x,y) - K(x,y)\|^2 = \langle R-K, R-K\rangle = \langle R-K, R\rangle - \langle R-K, K\rangle = 0$$
$$\Rightarrow R(x,y) = K(x,y), \forall x, y \in X.$$

Lemma 1.3: The reproducing kernel $R(x,y)$ is symmetric.

Proof: Note that $R_x(y) = \langle R_x(.), R_y(.)\rangle = \langle R_y(.), R_x(.)\rangle = R_y(x)$.

Lemma 1.4: The reproducing kernel $R(x,y)$ is positive semi-definite. That is, $R_x(x) \geq 0$, for any fixed $x \in [a,b]$.

Proof: Note that $R_x(x) = \langle R_x(.), R_x(.)\rangle = \|R_x(.)\|^2 \geq 0$.

Theorem 1.3: (Aronszajn, 1950) For a Hilbert space $\mathcal{H}$ of functions on X, there exists a reproducing kernel R for $\mathcal{H}$ if and only if for every $x \in X$, the evaluation linear functional $I: f \to f(x)$ is bounded linear functional.

Theorem 1.4: (Aronszajn, 1950) The reproducing kernel $R_x(y)$ of reproducing kernel of Hilbert space $\mathcal{H}$ is positive definite kernel.

Theorem 1.5: (Aronszajn, 1950) Every sequence of functions $(f_n)_{n\geq 1}$ which converges strongly to a function f in $\mathcal{H}_R(X)$, converges also in the point wise sense, that is, $\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in X$. Furthermore, this convergence is uniform on every subset of X on which $x \to R(x,x)$ is bounded.

Definition 1.16 : The function space $W_2^m[a,b]$ is defined by $W_2^m[a,b] = \{u: u^{(j)}$ is Abs. C, $j = 1,2,\dots,m-1$, and $u^{(m)} \in L^2[a,b]\}$. The inner product and the norm are given, by :

$$\langle u, v\rangle_{W_2^m} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx,$$

and $\|u\|_{W_2^m} = \sqrt{\langle u, u\rangle_{W_2^m}}$.

Theorem 1.6 : (Aronszajn, 1950) The function space $W_2^m[a,b]$ is called a reproducing kernel space if for each fixed $x \in [a,b]$ and for any $u(y) \in W_2^m[a,b]$, $\exists R_x(y) \in W_2^m[a,b]$, $y \in [a,b]$ such that $\langle u(y), R_x(y)\rangle = u(x)$, whereas $R_x(y)$ is called reproducing kernel function of the space $W_2^m[a,b]$.