LIST OF SYMBOLS

Number	Caption	
1	$\mathcal{H}$	Hilbert space
2	$W_2^3[0,1]$	Sobolev space of order 3
3	RKHS	Reproducing kernel Hilbert space
4	$\|\cdot\|_{W_2^3}$	Norm in the sense of the space $W_2^2[0,1]$
5	$\{\psi_i(x)\}_{i=1}^{\infty}$	Orthogonal Function System
6	β_{ik}^{s}	Orthogonalization Coefficients
7	$\overline{\psi_i}(x)$	Orthonormal Function
8	$\psi_i(x)$	Orthogonal Function
9	$\{\overline{\psi_i}(x)\}_{i=1}^{\infty}$	Orthonormal Function System
10	δ_x	Dirac functional
11	Abs. Cts.	Absolutely continuous
12	L^p	Lebesgue space
13	L_y	linear bounded operator applied on y
14	$\mathbb{I}$	The identity operator
15	$R_x(y)$	Reproducing kernel functions
16	$u_n(x)$	Approximated solution

REPRODUCING KERNEL HILBERT SPACE FOR CERTAIN CLASS OF SECOND BOUNDARY VALUE PROBLEM: THEORIES AND APPLICATIONS

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ABSTRACT

In this thesis, an efficient algorithm based on the reproducing kernel Hilbert space (RKHS) method is presented for solving a class of differential equations. The RKHS method has been applied for obtaining the approximated solutions for a class of second order differential equations. Meanwhile, the reproducing kernel function and its conjugate operator have been employed for constructing the complete orthonormal basis in the space $W_2^3[0,1]$. The analytical and approximated solution are represented in the form of a convergent series with accurately computable structures in the space $W_2^3[0,1]$. The n-th term approximation is proved to converge uniformly to the analytical solution. The main features of the RKHS method lie in that it can be directly applied for solving nonlinear problems without the need for unphysical restrictive assumptions, such as linearization, discretization, perturbation, or guessing the initial data. Further, the numerical comparison between the proposed method and the given exact solution is discussed by providing illustrated examples. The gained results reveal that the RKHS method is a systematic technique in obtaining accurate solutions for many nonlinear problems arising in natural sciences.

Introduction

Second-order boundary value problems (BVPs) for ordinary differential equations are encountered very often in applied mathematics, physics and engineering such as atomic calculations, gas dynamics, and so on (Chandrasekhar, 1981; Soedel, 1993; Na, 1979; Dulacska, 1992). Recently, nonlinear second-order periodic BVPs, which consist of second-order ordinary differential equations combined with periodic boundary conditions, have been vastly studied due to their broad range of application. (Atici and Guseinov, 2011; Li and Liang, 2005; Agarwal et al., 2006; Seda, 1992), But those BVPs do not always have solutions which can be obtained using analytical methods, and must be approached with various approximate and numerical methods.

The theory of reproducing kernel was put in use for the first time in the early 20th century as a solver for the BVPs of both harmonic and bi-harmonic functions. This theory has been effectively used as a base for constructing numerical solutions to applied sciences and various other important applications. (Saitoh, 1988; Daniel, 2003; Li and Cui, 2003). In the recent years, based on this theory, extensive work has been proposed and discussed for the numerical solutions of several integral and differential operators side by side with their theories. The researchers are generously asked to go through (Cui and Lin 2008; AL-Smadi, 2011; Geng, 2009; Ye and Geng, 2009; Komashynska and Al-Smadi, 2014; Abu Arqub et al., 2012; Abu Arqub et al., 2015; Altawallbeh et al., 2013) in order to know more details about the RKHS method, including its modification and scientific applications, its characteristics and symmetric kernel functions, and others.

The advantages of the proposed method include: accuracy with minimal effort needed to achieve the results, possibility of picking any point in the interval of integration and the approximate solutions and their derivatives will be applicable, the method does not require discretization of the variables and is not affected by computation round off errors and it is of global nature in terms of the solutions obtained.

In this thesis, the analytical approximated solution has been investigated using the reproducing kernel Hilbert space (RKHS) method for BVPs of second-order with non-classical conditions, and the analytic and approximate solutions are given in series form in the appropriate spaces. This thesis is arranged in the following order: In Chapter One, the basic mathematical concepts needed are introduced and a brief overview of the function's standard spaces of interest is given. Chapter Two gives a brief introduction on RKHS preliminaries relevant to this study. In Chapter Three, RKHS technique are applied to develop a numerical method in the Hilbert Space $W_2^3[0,1]$ used for the acquisition of approximations of the solution and its derivatives for the general form of second order differential equations, in addition to numerical experiments and simulation results. This work ends in Chapter Four with some concluding remarks.

CHAPTER ONE

PRELIMINARIES AND NOTATIONS

Chapter One

Preliminaries and Notations

In this chapter, we will introduce the notations and symbols, which will be used in the remainder of our thesis and give a brief overview of the function's standard spaces of interest. The symbols $\mathbb{R}$ and $\mathbb{C}$ indicate the set of real and complex numbers, respectively. We will use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. The j^{th} component of a vector x is denoted by x_j. The complex conjugate of a number z is $\bar{z}$, the absolute value is denoted by $|x|$. Throughout the whole thesis, $\{\psi_i\}_{i=1}^{\infty}$and $\{\overline{\psi_i}\}_{i=1}^{\infty}$ refer to the orthogonal and orthonormal function system, respectively. The Greek letters, α and β refer to real numbers. The capital Latin letters $\mathcal{H}$ and $\mathcal{W}$ refer to RKHS and the letters K, R and T are always used for its reproducing kernel function.

1.1 Basic Mathematical Concepts

The function fields possess the following axioms (the Field Axioms) (Adamson, 2007), for all elements $a, b, c \in \mathbb{F}$.

- Associative Laws:

  $a + (b + c) = (a + b) + c,$

  $a.(b.c) = (a, b).c,$

- Commutative Laws

  $a + b = b + a.$

  $a.b = b.a,$

- Identity Law

  $a + 0 = 0 + a = a.1 = 1.a = a.$

- Distributive Law

  $a.(b + c) = a.b + a.c,$

  $(a + b).c = a.c + b.c.$

- Inverse Law

  $a + (-a) = (-a) + a = 0$, $a \, . \, a^{-1} = 1$, when $a \neq 0$.

Further, the notion of isomorphism is very commonly used in all areas of math. It is basically the mapping between two objects that maintains all the relevant properties of said objects. In the case of fields $\mathbb{H}$ and $\mathbb{G}$, we say that ϕ is an isomorphism between $\mathbb{H}$ and $\mathbb{G}$ if ϕ is a function from $\mathbb{H}$ to $\mathbb{G}$ and ϕ obeys certain properties (Awodey, 2006):

- Injective (one to one): for all $, f' \in \mathbb{H}$, $\phi(f) = \phi(f')$ implies that $f = f'$.
- Surjective (onto): for all $g \in \mathbb{G}$ there exists $f \in \mathbb{H}$ such that $\phi(f) = g$.
- Preservation: essentially, ϕ preserves operations. That is for example, $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$.

A vector space X over a field $\mathbb{F}$ is a collection of objects with a "*vector*" addition and scalar multiplication, and is defined as closed under both operations and which in addition satisfies the following axioms:

1. $(\alpha + \beta)x = \alpha x + \beta x\ forall x \in X\ and\ \alpha, \beta \in \mathbb{F}$.
2. $\alpha (\beta x) = (\alpha\beta) x$.
3. $(x + y) = (y + x)$, for all $x , y \in X$.
4. $x + (y + z) = (x + y) + z$, for all x , y and $z \in X$.
5. $\alpha (x + y) = \alpha x + \alpha y$, for all $x , y \in X$ and $\alpha \in \mathbb{F}$.
6. $\exists 0 \in X\ such\ that\ 0 + x = x$ for all $x \in X$.
7. For each $x \in X$ there exists a vector $u \in X$ such that $x + u = u + x = 0$.

Definition 1.1 A vector space X is called an inner product space if there is a mapping of $X \times X$ into a field F that satisfies the following properties:

1) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,\ \forall x, y, z \in X$.
2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle,\ \forall x, y \in X,\ \forall \alpha \in \mathbb{F}$.
3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
4) $\langle x, x\rangle \geq 0$, with equality iff $x = 0$

Remark 1.1 On $\mathbb{C}^n$, we have the standard inner product which defined by

$\langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$, where $x = (x_1, \dots., x_n) \in \mathbb{C}$ and $y = (y_{1,\dots\dots,} y_n) \in \mathbb{C}$

Remark 1.2 On $\mathbb{R}^n$, we have the standard inner product which is defined by

$\langle x, y\rangle = \sum_{i=1}^{n} x_i \, y_i$, where $x = (x_1, \dots., x_n) \in \mathbb{R}$ and $y = (y_{1,\dots\dots,} y_n) \in \mathbb{R}$

Remark 1.3 A vector space together with the inner product is called an inner product space.

Definition 1.2 A normed space E is a vector space over a field F with a norm defined on it such that $\| . \| : E \to [0, \infty)$ satisfies the following:

1) $\|a\| \geq 0$ with equality iff $a = 0$.
2) $\|\alpha a\| = |\alpha|\|a\|, \ \forall \alpha \in$ F.
3) $\|a + b\| \leq \|a\| + \|b\|, \ \forall a, b \in E$.

Note that $\forall a \in E, \ \|a\| = \sqrt{\langle a, a\rangle}$.

The following linking equations between the norm and inner product hold:

• $|\langle a, b\rangle| \leq \|a\| . \|b\|$ (Cauchy-Schwarz inequality)

• $\|a + b\|^2 + \|a - b\|^2 = 2\|a\|^2 + 2\|b\|^2$ (the parallelogram law)

Remark 1.4 A vector space together with the norm function is called normed vector space.

Definition 1.3 A sequence $\{x_n\}_{n=1}^{\infty}$in a normed space X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$. x is called the limit of (x_n) and we write $\lim_{n\to\infty} x_n = x$, simply, $x_n \to$ x . We say that (x_n) converges to x.

Definition 1.4 A sequence $\{x_n\}_{n=1}^{\infty}$in a normed space X is called a Cauchy sequence if $\forall \varepsilon > 0$, there is a positive real number N_ε such that $\|x_n - x_m\| < \epsilon, \forall n, m > N_\epsilon$.

Remark 1.5 Every convergent sequence is Cauchy convergent, but Cauchy sequence is not always converge. The space where every Cauchy converge is convergent to some point in the spaces is called complete space; complete normed vector space is called Banach space that is called a complete normed space.

Definition 1.5 A Hilbert space H is an inner product space in which every Cauchy sequence in H converges to some element in H, which is called a complete inner product space.

Remark 1.6 The normed space X is called complete if every Cauchy sequence in X converges: it has a limit in X. However, every Hilbert space is Banach space with respect to the norm $\|x\| = \sqrt{\langle x, x\rangle}$.

Definition 1.6 let x and y be vectors of an inner product space H, if their product is zero, then they are orthogonal. $(x \perp y)$.

Definition 1.7 Let K, S be two normed spaces Q map $Q: K \to S$ is called a linear operator if

$$Q(\alpha t + \beta r) = \alpha Q(t) + \beta Q(r), \qquad \forall \alpha, \beta \in \mathbb{R}, \qquad \forall t, r \in K.$$

If $S = \mathbb{R}$, then Q is called linear functional.

Definition 1.8 Let K, S be two normed spaces and $Q: D(Q) \subset K \to S$ be a linear operator. Then Q is called

- Bounded linear operator if $\exists \lambda > 0$ such that $\|Q_x\| \leq \lambda \|x\|$, $\forall x \in D(Q)$. the smallest value of λ is called the operators norm and denoted by $\| Q \|$, and linear operator is called functional.
- $Q^*: \mathcal{H}_2 \to \mathcal{H}_1$ is called the adjoint operator for $\mathbb{T}$ $if (Q(x), y) = (x, Q^*(y))$ where$x, y \in \mathcal{H}_2$, in special case if $Q = Q^*$ we say Q is self-adjoint (symmetric) operator.
- Q is called unitary (orthogonal) if $QQ^* = Q^*Q = \mathbb{I}$, where $\mathbb{I}$ is the identity operator.

Definition 1.9 Let $Q: \mathcal{D}(Q) \longrightarrow S$ be any operator, not necessarily linear, where $\mathcal{D}(Q) \subset X$ and X, S are two normed spaces, then the operator Q is said to be continuous at $x_0 \in \mathcal{D}(Q)$,if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Q_x - Q_{x_0}\| < \varepsilon$ for all $x \in \mathcal{D}(Q)$ satisfying $\|x - x_0\| < \delta$.

Remark 1.7 Q is continuous if Q is continuous at every $x \in \mathcal{D}(Q)$.

Definition 1.10 Let $Q: \mathcal{D}(\mathrm{Q}) \longrightarrow \mathrm{S}$ be a linear operator, where $D(Q) \subset X$ and X, S are normed space. Then, Q is continuous if and only if Q is bounded

Definition 1.11 Let $Q: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. Then, the Hilbert- adjoint operator Q^* of Q is the operator $Q^*: \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, $\langle Qx, y\rangle = \langle x, Q^* y\rangle$**.** Whereas Q is self -adjoint if $Q = Q^*$.

For example, three orthogonal functional $\boldsymbol{T}_a, \boldsymbol{E}_b, \boldsymbol{D}_c$ can be define on $L^2(\mathbb{R})$ (Derezinski, 2007):

- Translation operator: For $a \in \mathbb{R}$, we define the translation operator $\boldsymbol{T}_a: L^2(\mathbb{R}) \to L^2(\mathbb{R})$, by $(\boldsymbol{T}_a\, f)(x) := f(x-a)$, where $f(x) \in L^2(\mathbb{R})$.

- Modulation operator: For $b \in \mathbb{R}$, we define the modulation operator $\boldsymbol{E}_b: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $\boldsymbol{E}_b(x) := e^{\pi ibx} f(x)$**.**

- Dilation operator: For $c \in \mathbb{R}$, we define the dilation operator $\boldsymbol{D}_c: L^2(\mathbb{R}) \to L^2(\mathbb{R})$,by $(\boldsymbol{D}_c\, f)(x) := f(x/c)/\sqrt{c}$.

Definition 1.12 Let *H* is a Hilbert space of functions *f*: *X* → ℝ, defined on a non-empty set *X*. For a fixed $x \in X$, the map $\delta_x: \mathcal{H} \to \mathbb{R}$, such that $\delta_x(f) = f(x)$ is said to be the Dirac evaluation at x.

Evaluation functional is always linear: For $f, g \in \mathcal{H}$ and or $\alpha, \beta \in \mathbb{R}$, $\delta_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$, and we say that δ_x is bounded if There exists $k \geq 0$ such that $\| \delta_x(f) \| \mathbb{R} \leq k \| f \| \mathcal{H}, \forall f \in \mathcal{H}$. The delta function δ_x has the property (Al-Smadi, 2011):

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a),$$

where the delta function is define as:

$$\delta(x-y) = \begin{cases} 0, & x \neq y \\ \infty, & x = y \end{cases} .$$

Theorem 1.1 (Riesz's Theorem) Every bounded linear functional A on a Hilbert space $\mathcal{H}$ can be represented in terms of the inner product, namely $A(x) = \langle x, y \rangle$, where y depends on A, is uniquely determined by A and has norm the $\|y\| = \|A\|$. (There is a unique $k \in \mathcal{H}_1$ such that

$$\Phi(xv) = (x.k)_{\mathcal{H}_1}, \forall x \in \mathcal{H}_1.$$

Definition 1.13 A function $Q: [a, b] \longrightarrow \mathbb{R}$ is called absolutely continuous, if for every positive ε, there exists a positive δ such that $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ $(\subset [a, b])$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$, then $\sum_{i=1}^{n} |Q(y_i) - Q(x_i)| < \varepsilon$, where $[a , b]$ disjoint.

Definition 1.14 Let Φ be a domain in $\mathbb{R}^n$ where $1 \leq p < \infty$. The class of all measurable functions u defined on Φ for which $\int_\Phi |u(x)|^p dx < \infty$ is denoted by $L^p(\Phi)$, which is a Banach space with respect to the norms $\|u\|_{L^p} = \left(\int_\Phi |u(x)|^p dx\right)^{1/p}$.

1.2 Inner Product Spaces

In the present subsection, we will review some spaces and their norms to understand the nature of spaces construction:

1. Lebesgue Spaces

- We refer to $L^P(\Omega)$, where $1 \leq P < \infty$, as the linear space of $p - th$ order Integrable functions on Ω. In general, for a measurable set Ω, and $1 \leq P < \infty$, we say that $f \in L^P(\Omega)$, if

1) f is measurable real valued function.
2) $\int_\Omega |f(x)|^p dx < \infty$.

It worth's to mention here that $L^P(\Omega)$ form a Banch space with respect to the norm $\|u\|_{L^P} := \left(\int_\Omega |f(x)|^p dx\right)^{1/p}$ is finite. For $p = \infty$, we refer to $L^\infty(\Omega)$ as the linear space of basically functions, where $f \in L^\infty(\Omega)$ if f is measurable with $\|u\|_{L^\infty} < \infty$. $L^\infty(\Omega)$ is a