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Permutable Subgroups of Groups of Order 16

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Declaration

I certify that all the material in this thesis that is well referenced and its contents is not submitted, in any other place, to get another certificate.

The contents of this thesis reflect my own personal views, and are not necessarily endorsed by the University.

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Committee Decision

We certify that we have read the present work and that in our opinion it is fully adequate in scope and quality as thesis towards the partial fulfillment of the Master Degree requirements in

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Dedication

I would like to dedicate my work to the soul of my parents.

Acknowledgment

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Abstract

A permutable (quasinormal) subgroup K of a group G is a subgroup which permutes with every subgroup of G i.e. $HK = KH$ for every subgroup H of G. This sort of subgroups has certain properties, such as these subgroups are always subnormal. Certainly, every normal subgroup is permutable, the converse is not always true. The main interest in this thesis is to find all permutable subgroups of a group of order 16, in addition, to classify which of these subgroups is permutable and not normal. Therefore, this thesis will give some examples of permutable subgroup which is not normal.

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List Of Abbreviations

$A \setminus B$	$\{x \in A \mid x \notin B\}$.
$\mathrm{Aut}(G)$	The automorphism group of a group G.
$C_G(x)$	The centralizer of x in G.
$\mathbb{Z}_n$	The cyclic group of order n.
D_n	The dihedral group of order $2n$.
G'	The derived subgroup of the group G.
G/N	The factor group of G by N, or quotient group of G over N.
$\gcd(a, b)$	The greatest common divisor of the integers a and b.
$H \sim G$	H is isomorphic to G.
$H < G$	H is a proper subgroup of G.
$H \leq G$	H is a subgroup of G.
$H \rtimes K$	The semidirect product of the normal subgroup H and K.
$H \times K$	The direct product of H and K.
$H \trianglelefteq G$	H is a normal subgroup of G.
$i \mid n$	i divides n.
$i \nmid n$	i does not divide n.
$\mathrm{lcm}(a, b)$	The least common multiple of a and b.
$\mathbb{N}$	The set of positive integers.
$Z(G)$	The center of the group G.
$\lvert G \rvert$	The Number of elements in the set G.
$[G : H] = \frac{\lvert G \rvert}{\lvert H \rvert}$	The index of the subgroup H in the group G.
U_n	The group of units in $\mathbb{Z}_n$

Chapter 1

Introduction

In group theory, we often interested in classifying all groups of a certain order by isomorphism class, which demonstrates that they have the same structure. This thesis focus mainly on classify and organize finite permutable subgroups of groups of order 16.

1.1. Literature Review

A constantly theme in group theory has a major roll in classifying all groups of proper subgroups with some given property. R. Dedekind **[1]** started the base idea of this type of research, he classified the finite groups in which all subgroups are normal. This theme has been continued using several generalizations of normality. One such generalization is the notion of subnormality. The literature concerning groups with all subnormal subgroups is long and varied, one highlight is the theorem of Mohres **[2]** such that a group is soluble. Further details concerning groups with all subgroups are subnormal can be found in **[3]** and **[2]**. Another generalization of normality is permutability. A subgroup H of a group G is said to be permutable if $HK = KH$ for every subgroup K of G, in **[4]** the structure of infinite groups with all subgroups permutable was obtained. In **[5]**, he extend the classification to the groups of order 16, this was done by representing the groups using the generators and the orders of these generators, also the group centre take a place in this representation. In particular, the representation found in **[5]** is used frequently in this thesis.

1.2. Thesis Objectives

The main topic in this research is permutable subgroups of groups of order 16. This thesis is intend to:

(1) Classifying all the considered groups by isomorphism class.
(2) Classify and organize the permutable subgroups of the considered groups.
(3) State and show an example of permutable subgroup which is not normal.
(4) Using `GAP` to discuss our results.

1.3. Thesis Organization

This research consists of five chapters.

Chapter One, is the introduction of this research. Which includes, some of the earlier studies, the main objectives, and the research organization.

In Chapter Two, we give some preliminaries, some basic concepts and elementary results in group theory that is necessary for this thesis.

Chapter Three, considers some main topics in group theory, such as; automorphism of groups, the semidirect product and some examples of semidirect product.

Next, in Chapter Four, we use David Clausen method [**5**] to classify the groups of order 16, which based on considering the different cases for the order of the center. Also, we use `GAP` to help us in our studying and to give a specific information about permutable subgroup structure of groups of order 16 especially the non-abelian groups.

In Chapter Five we give examples of subgroups which are permutable but not normal, which declare the main aim of this thesis.

In the last chapter, the conclusions of this research were concluded and discussed.

Preliminaries

2.1. Introduction

In this part we introduce some of the basic concepts and elementary results in group theory that is necessary for this thesis.

All considered groups in this thesis are finite. Also, Our notations and basic concepts are standard and consistent with the notations in **[6]**, **[3]** and **[7]**.

2.2. Some basic concepts

The main topic in this thesis is the group. The next definitions and results describe this algebraic structure as well.

Definition 2.2.1. **[6]** A binary operation $*$ on a non-empty set S is a rule that assigns to each ordered pair $(a, b) \in S \times S$ an element $x \in S$. This can be written as $*(a, b) = x$ or $a * b = x$.

The set S endowed with the binary operation $*$ will be written $\langle S, *\rangle$.

Definition 2.2.2. **[6]** A group $\langle G, *\rangle$ is a set G which is closed under the binary operation $*$, such that the following axioms are hold:

(1) The binary operation $*$ is associative.
(2) There is an element e in G such that $e * g = g * e = g$, $\forall g \in G$ (The element e is called the identity element of $*$ on G)
(3) For each a in G, there is an element a^{-1} in G with the property that $a^{-1} * a = a * a^{-1} = e$. (The element a^{-1} is the inverse of a with respect to the operation $*$ and is denoted by a^{-1}).

Theorem 2.2.1. **[6]** *A subset H of a group G is a subgroup of G if and only if:*

(1) $x\, y \in H$ *for all* $x, y \in H$.
(2) $e \in H$.
(3) $x^{-1} \in H$ *for all* $x \in H$.

Proof. See **[6]**. □

Proposition 2.2.1. **[8]** *A non-empty subset H of a group G is a subgroup of G if and only if $x\, y^{-1} \in H$ whenever $x, y \in H$.*

Proof. See [8] □

Definition 2.2.3. [9] A group G is abelian if it is binary operation is commutative.

Definition 2.2.4. [6] Let G be a group and let $a \in G$.

- The group $H = \{a^n \mid n \in \mathbb{Z}\}$ is the cyclic subgroup of G generated by a, and denoted by $\langle a \rangle$.
- An element a of a group G generates G and is a generator for G if $\langle a \rangle = G$.
- A group G is cyclic if there is some element in G that generates G.
- Let M be an arbitrary nonempty subset of a group G. The intersection of all subgroups of G which contain M is said to be the subgroup generated by M and is denoted by $\langle M \rangle$.

Definition 2.2.5. [6] A maximal subgroup of a group G is a subgroup M not equal to G such that there is no proper subgroup N of G properly containing M. We will denote the maximal subgroup M of G by $M < .G$.

Definition 2.2.6. [6] A subgroup N of a group G is called a normal subgroup of G if $Na = aN$ for all $a \in G$ and is denoted by $N \trianglelefteq G$. Where $aN = \{an \mid n \in N\}$. The set aN is called the left coset of N by a.

Definition 2.2.7. [6] Let H be subgroup of a group G. The number of distinct left cosets of H in G is the index $(G : H)$ of H in G. Since G is finite, then $(G : H) = \dfrac{|G|}{|H|}$, because every cosets of H contains $|H|$ elements.

Definition 2.2.8. [6] Let N be a normal subgroup of a group G. Then the set of all cosets of N in G together with the binary operation of cosets multiplication is called the factor group (or quotient group) of G by N and it is denoted by G/N.

Definition 2.2.9. [6] The center of a group G is the subgroup $\{g \in G \mid gx = xg, \forall x \in G\}$ of G and denoted by $Z(G)$.

Theorem 2.2.2. [9] *The center of a group is a normal subgroup.*

Theorem 2.2.3. [9] *If $G/Z(G)$ is cyclic, then G is abelian group.*

One can easily shows that every cyclic group G is abelian.

Example 2.2.1. The set $G = \{0, 1, 2, \cdots, n-1\}$ with addition modulo n is a cyclic group. This group will be denoted by $\mathbb{Z}_n$.

Example 2.2.2. The group $G = \{e, a, b, c\}$ endowed with a binary operation that satisfies: e is the identity element and $x^2 = e$ for all $x \in G$. This group is called the Klein 4-group. Certainly, this group is abelian but not cyclic.

Definition 2.2.10. [6] Let G and H be two groups. The direct product of G and H is the group $\{(g, h) \mid g \in G\ ,\ h \in H\}$ endowed with the binary operation $*$ defined by $(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$, where $*_G$ and $*_H$ are the groups operations of G and H respectively.

Definition 2.2.11. [6] A map ϕ from a group G into a group H is a homomorphism if $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in G$.

Definition 2.2.12. [6] A bijection homomorphism $\phi : G \to H$ is called an isomorphism. The groups G and H are said to be isomorphic, and denoted by $G \sim H$.

Theorem 2.2.4. **[9]** *Let H and N be subgroups of G and $N \trianglelefteq G$. Then $H/H \cap N \sim HN/N$.*

Theorem 2.2.5. **[9]** *Let G be a group and let H and K be normal subgroups of G with $H \trianglelefteq K$. Then*

$$(G/H)/(K/H) \sim G/K.$$

Definition 2.2.13. **[6]** A group G is a p-group if every element in G has order a power of the prime p. A subgroup of a group G is a p-subgroup of G if the subgroup itself is a p-group.

Theorem 2.2.6. **[9]** *For a p-group G, The center of the group is a nontrivial subgroup. i.e $|Z(G)| \geq 2$.*

Theorem 2.2.7. **[6]** *Let H be subgroup of a finite group G. Then the order of H divides the order of G.*

This result is a direct consequence of Lagrange's theorem. It should be remarked that the converse of Lagrange theorem is not true.

Theorem 2.2.8. **[6]***(Cauchy's theorem)*
Given a finite group G and a prime number p dividing the order of G, then there exists an element (and hence a cyclic subgroup) of order p in G.

Theorem 2.2.9. **[6]** *[First Sylow theorem]For every prime factor p with multiplicity n of the order of a finite group G, there exists a Sylow p-subgroup of G, of order p^n.*

Theorem 2.2.10. **[9]***(Fundamental Theorem of Finite Abelian Groups)*
Every finite abelian group is isomorphic to a direct product of cyclic groups of the form $\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_n^{\alpha_n}}$, where the p_i are (not necessarily distinct) primes.

Definition 2.2.14. **[10]** A subgroup of a group is termed permutable (or quasinormal) if it satisfies the following equivalent conditions:

(1) Its product with every subgroup of the group is a subgroup.

(2) It permutes (or commutes) with every subgroup.

(3) It permutes with every cyclic subgroup.

Definition 2.2.15. **[6]** Let X be a set and G a group. An action of G on X is a mapping $* : GH \to H$, such that:

- $*(e, x) = e * x = x$ for all $x \in X$,
- $*(g_1g_2, x) = (g_1g_2) * x = (g_1g_2)x = g_1(g_2x)$, for all $x \in X$ and all $g_1, g_2 \in G$.

Under these conditions X is called a G-set.

Groups of order 16 as an example of semidirect product

3.1. Introduction

In this part we study automorphism of a group and all of its relevant properties and we will provide some theorems and corollaries about it. Then we will discuss the semidirect product to classify groups of order 16.

3.2. Automorphism

Definition 3.2.1. **[7]** Let G be a group an isomorphism from G onto itself is called an automorphism of G. The set of all automorphism of G is denoted by $Aut(G)$.

Example 3.2.1.
- Consider the automorphism $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = \bar{z}$, i.e. $\phi(x + iy) = x - iy$. This is an automorphism and ϕ only fixed for $z = 0$.
- The function $\phi(x) = -x$ is an automorphism in $\mathbb{Z}$.

Theorem 3.2.1. **[8]** *Let G be a group, then $Aut(G)$ is a group under the operation of composition for functions.*

Proof. Let G be a group. Then $Aut(G) = \{F : G \to G \mid F \text{is an isomorphism} of G\}$. Given two automorphisms $f, g \in Aut(G)$, we can consider the composition $g \circ f$. We claim that $(Aut(G), \circ)$ is a group. Since g and f are bijective, then $g \circ f$ is bijective. Moreover, for all $a, b \in G$, it follows:

$$\begin{aligned}
(g \circ f)_{(ab)} &= g(f(ab)) = g(f(a)f(b)) \\
&= g(f(a))g(f(b)) \\
&= (g \circ f)_{(a)}(g \circ f)_{(b)} \\
&= g(f(ab)) = g(f(a)f(b)) \\
&= g(f(a))g(f(b)) \\
&= (g \circ f)_{(a)}(g \circ f)_{(b)}
\end{aligned}$$

Hence $g \circ f$ is a group homomorphism. Secondly we need to show that $(h \circ g) \circ f = h \circ (g \circ f)$. It is enough to evaluate both morphisms at $a \in G$ and see that both expressions coincide due to the associativity of G. Next, we need to check that there is an identity element for $\circ$. Clearly $Id_G : G \to G$ defined by $Id_G(a) = a$ for all $a \in G$ is an automorphism. Clearly, $f \circ Id_G = Id_G \circ f$,for all $f \in Aut(G)$. Thus Id_G is the identity element. Finally, let $f \in Aut(G)$, then the inverse function f^{-1} is will defined and $f^{-1} \in Aut(G)$. Clearly $f^{-1} \circ f = Id_G = f \circ f^{-1}$. □

Proposition 3.2.1. **[7]** *The Automorphism group of the cyclic group of order n isomorphic to U_n.*

Proof. Let x be a generator of the cyclic group $\mathbb{Z}_n$ if $\psi \in Aut(\mathbb{Z}_n)$, then $\psi(x) = x^a$ for some $a \in \mathbb{Z}$ and the integer a uniquely determines ψ. Denote this automorphism by ψ_a. As usual, since $|x| = n$, the integer a is only defined $\mod n$. Since ψ is an automorphism, x and x^a must have the same order, hence $(a, n) = 1$. Further, for every a relatively prime to n, the map $x \to x^a$ is an automorphism of $\mathbb{Z}_n$. Hence we have a surjective map

$$\psi : Aut(\mathbb{Z}_n) \to U_n \ , \ where\ U_n \equiv (\mathbb{Z}/n\mathbb{Z})^\times$$

$$\psi_a \to a \mod n$$

The map ψ is a homomorphism because

$$(\psi_a \circ \psi_b)_{(x)} = \psi_a(x^b) = (x^b)^a = x^{ab} = \psi_{ab}(x) \ , \ \forall \psi_a, \psi_b \in Aut(\mathbb{Z}_n).$$

So that

$$\psi(\psi_a \circ \psi_b)) = \psi(\psi_{ab}) = ab \mod n = \psi(\psi_a)\psi(\psi_b).$$

Finally , ψ is clearly injective. Hence it is an isomorphism. □

Proposition 3.2.2. **[7]** *$Aut(\mathbb{Z}_p) \sim \mathbb{Z}_{p-1}$, (p prime).*

Proof. We know that $Aut(\mathbb{Z}_p) \sim U(p)$ by Proposition 3.2.1. By Gauss's result, we know that when p is prime, then $U(p^k) \sim \mathbb{Z}_{\phi(p^k)}$ where $\phi(n)$ is Euler phi-function. Hence, $U(p) \sim \mathbb{Z}_{p-1}$, and isomorphism follows transitivity laws. So, $Aut(\mathbb{Z}_p) \sim \mathbb{Z}_{p-1}$ when p is prime. □

Example 3.2.2. For $Aut(\mathbb{Z}_{10}) \simeq U_{10} = \{1, 3, 7, 9\}$, there are only two groups of order 4, which are, $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Notice that $3^2 = 9 \mod 10$. Since 3 does not have order 2, it follows that $Aut(\mathbb{Z}_{10})$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, $Aut(\mathbb{Z}_{10}) \simeq \mathbb{Z}_4$.

Proposition 3.2.3. **[7]** *For $n \geq 3$. The automorphism group of the cyclic group of order 2^n is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$. In particular not cyclic but has a cyclic subgroup of Index 2.*

Let G be a group of order p^2, Then either G is cyclic and $|Aut(G)| = p(p-1)$ or G is elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, $|Aut(G)| = p(p-1)^2(p+1)$.

3.3. Semidirect Products

Theorem 3.3.1. **[7]** *Let H and K be groups and let ϕ be a homomorphism from K into $Aut(H)$. Let $\cdot$ denotes the (left) action of K on H determined by ϕ. Let G be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$ and define a multiplication $*$ on G by:*

$$(h_1, k_1) * (h_2, k_2) = (h_1 k_1 \cdot h_2, k_1 k_2)$$

(1) *This multiplication makes G into a group of order $|G| = |H||K|$.*
(2) *The sets $\{(h, e_K) \mid h \in H\}$ and $\{(e_H, k) \mid k \in K\}$ are subgroups of G and the maps $h \to (h, e_K)$ for $h \in H$ and $k \to (e_G, k)$ for $k \in K$ are isomorphisms of these subgroups with the group H and K respectively.*